Kähler manifolds

particular kinds of complex manifolds with additional structure.

$\boxed{\text{Example}}$ $\mathbb{R}^{2n} = \{(x_1, y_1, x_2, y_2, \dots, x_n, y_n) : \begin{smallmatrix} x_j \in \mathbb{R} \\ y_j \in \mathbb{R} \end{smallmatrix}\ \forall j\}$

Complex Structure At each point p of $\mathbb{R}^{2n}$,

The tangent space $T_p\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$ ← I'll use the same coordinates for the vectors.

A linear map on $T_p\mathbb{R}^{2n}$ $(\partial_{x_1}, \partial_{y_1}, \dots, \partial_{x_n}, \partial_{y_n})$

An almost complex structure

$J : T_p\mathbb{R}^{2n} \to T_p\mathbb{R}^{2n}$ a linear transformation

with matrix
$$\begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}$$

$$J(\partial_{x_1}) = \partial_{y_1}$$
$$J(\partial_{y_1}) = -\partial_{x_1}$$
$$J^2 = -I$$

complexified tangent space. $T_p\mathbb{R}^{2n} \otimes \mathbb{C}$:

eigenvalues of J are $\pm i$

The $+i$ eigenspace is spanned by $\{\partial_{z_1}, \partial_{z_2}, \dots, \partial_{z_n}\}$

where $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$

(in complex analysis $\frac{\partial}{\partial z} = \partial_z = \frac{1}{2}\left(\underset{\uparrow \partial_x}{\frac{\partial}{\partial x}} - i\underset{\uparrow \partial_y}{\frac{\partial}{\partial y}}\right)$)

The $-i$ eigenspace is spanned

$\{\overline{\partial_{z_j}} : 1 \le j \le n\}$.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} +i \\ 1 \end{pmatrix} = i\begin{pmatrix} 1 \\ -i \end{pmatrix}. \checkmark$$

$T_p^{1,0}\mathbb{R}^{2n} = \operatorname{span}_{\mathbb{C}}\{\partial_{z_j} : 1 \le j \le n\}$ holomorphic tangent space

$T_p^{0,1}\mathbb{R}^{2n} = \operatorname{span}_{\mathbb{C}}\{\overline{\partial}_{z_j} : 1 \le j \le n\}$ antihol. tangent space

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Differential forms on $\mathbb{R}^{2n}$ can be written in terms of dx_j's & dy_j's or

$$dz_j = dx_j + i\,dy_j$$

$\left(dz_j(\partial_{z_k}) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases} \right.$

$$\frac{1}{2}(dz_j + d\overline{z}_j) = dx_j$$

$$\frac{1}{2i}(dz_j - d\overline{z}_j) = dy_j$$

So we can write all differential forms (with complex coeffs) in terms of dz_j's & $d\overline{z}_j$'s.

Exterior derivative

$$d : \underbrace{\Omega^*(\mathbb{R}^{2n})}_{\text{(complex) differential forms on } \mathbb{R}^{2n}} \longrightarrow \Omega^{*+1}(\mathbb{R}^{2n})$$

$$d = \sum dx_j \wedge \frac{\partial}{\partial x_j} + \sum dy_j \wedge \frac{\partial}{\partial y_j}$$

$$= \underbrace{\sum dz_j \wedge \frac{\partial}{\partial z_j}}_{\partial} + \underbrace{\sum d\overline{z}_j \wedge \frac{\partial}{\partial \overline{z}_j}}_{\overline{\partial}}$$

$d = \partial + \bar\partial \qquad V, W \in T_p\mathbb{R}^{2n}$

metric $g(V,W) = \langle V, W\rangle \underset{\mathbb{R}^{2n}}{=} V\cdot W = W\cdot V$

symmetric

fundamental form (2-form) $\omega(V,W) = g(JV, W)$

how does a 2-form eat two vector fields & get a fcn?

$(dx\wedge dw)(a\partial_x + b\partial_y + c\partial_w,\ d\partial_x + e\partial_y + f\partial_w)$

$= dx(a\partial_x + b\partial_y + c\partial_w)\, dw(d\partial_x + e\partial_y + f\partial_w)$

$- dw(a\partial_x + b\partial_y + c\partial_w)\, dx(d\partial_x + e\partial_y + f\partial_w)$

$= a\cdot f - c\cdot d.$

Notice $g(JW, V) = g(V, JW) = g(IV, JW)$

$= -g(J^2V, JW)$

J is compatible with the metric g

$\Leftrightarrow g(JV, JW) = g(V, W)$

$\Leftrightarrow \mathbb{R}^{2n}$: J is an orthogonal matrix.

$\longrightarrow g(JW, V) = -g(JV, W).$

$\omega(\cdot,\cdot)$ is antisymmetric $\Rightarrow$ a 2-form.

recall $J(\partial_{x_j}) = \partial_{y_j},\ J(\partial_{y_j}) = -\partial_{x_j}$

$$\omega(\partial_{x_j}, \partial_{y_j}) = g(J(\partial_{x_j}), \partial_{y_j}) = g(\partial_{y_j}, \partial_{y_j}) = 1$$

$$\omega(\partial_{x_j}, \underset{\text{else}}{\text{anything}}) = 0$$

$$\omega(\partial_{y_j}, \partial_{x_j}) = -1$$

$$\Rightarrow \boxed{\omega = \sum_{j=1}^{n} dx_j \wedge dy_j}$$ Fundamental form.

$$\boxed{\omega = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j}$$

in general, if $g(\cdot,\cdot)$ is a metric on $2n$ coordinates, you get

$$\omega = \frac{i}{2} \sum_{i,j=1} h_{ij}\, dz_i \wedge d\bar{z}_j$$

where h_{ij} is a Hermitian symmetric matrix at each point.

$$\left(h^* = \overline{(h^T)} = h \right)$$

Notice $\omega^n = \underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{n \text{ copies}} = n!\, \underbrace{dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n}_{\text{volume form on } \mathbb{R}^{2n}}$.

$$T_p^{1,0} = \operatorname{span}_{\mathbb{C}}\{\partial_{z_j}\} \quad T_p^{0,1} = \operatorname{span}_{\mathbb{C}}\{\partial_{\bar{z}_j}\}.$$

J is called an <u>integrable almost complex structure</u>

$=$ <u>complex structure</u>

if $T^{1,0}$ & $T^{1,0}$ are <u>involutive</u> subbundles of $TM \otimes \mathbb{C}$.

$\hookrightarrow$ If X, Y are two vector fields in $\Gamma(T^{1,0})$, then $[X,Y]$ is also in $\Gamma(T^{1,0})$.

Our J is integrable/involutive $\Rightarrow$ we have a complex structure.

FYI: This condition is satisfied iff

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] = 0$$

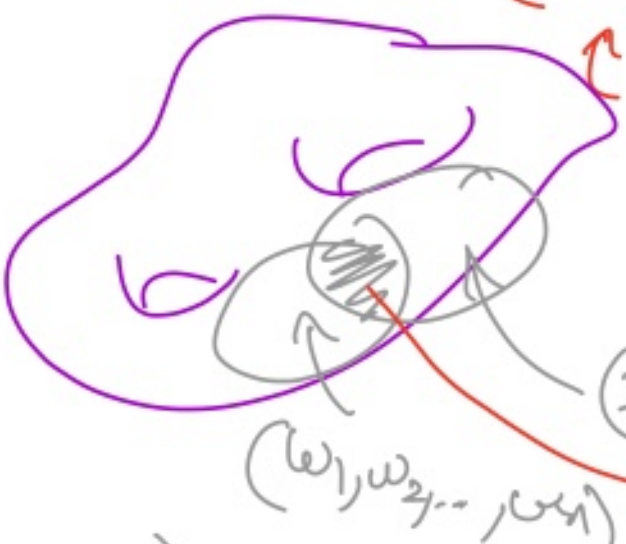
$\forall$ v.fields X, Y.

A complex manifold is

$$(M^{2n}, J)$$

complex structure

we can always find coordinates $(z_1, z_2, \dots, z_n)$ locally.



in overlap.

$\phi : \{(z_1, \dots, z_n)\} \to \{(w_1, \dots, w_n)\}$

is a biholomorphic map.

(M^{2n}, g, J) Hermitian manifold.

Kähler manifold:

$\underbrace{\underbrace{(M, J)}_{\text{complex}} + g \text{ compatible}}_{\text{Hermitian}} \Longrightarrow \omega$ fundamental

$+ \boxed{d\omega = 0}$ Kähler condition

ω Kähler form.

(M^{2n}, J, g, ω)

J ← almost cpx str. (integrable)

$$TM \otimes \mathbb{C} = T^{0,1} \oplus T^{1,0}$$

antihom. $\sum a_j \partial_{\bar{z}_j}$ holom. $\sum b_j \partial_{z_j}$

dual 1-forms dz_j, $d\bar{z}_j$

Differential forms $\Omega^k = \Omega^k(M)$

$= \{$differential k-forms$\}$

We can decompose

$$\Omega^k = \bigoplus_{r+s=k} \Omega^{r,s}, \text{ where}$$

$$\Omega^{r,s} = \sum u\, dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s}$$

We can decompose $d : \Omega^k \to \Omega^{k+1}$ exterior derivative

as $d = \underbrace{d^{1,0}}_{\partial} + \underbrace{d^{0,1}}_{\bar{\partial}} + \underbrace{d^{2,-1} + d^{-1,2}}_{0 \text{ on complex manifold.}}$,

$$d^{a,b} : \Omega^k \to \Omega^{k+1}$$

such that $\Omega^{r,s} \xrightarrow{d^{a,b}} \Omega^{r+a,\, b+s}$

Recall $\omega =$ fundamental 2-form $= \frac{i}{2} h_{ij}\, dz_i \wedge d\bar{z}_j$

$\omega \in \Omega^{1,1}$ Also $\omega = \bar{\omega}$

[$\mathbb{R}^{2n}$ $\omega = \sum dx_i \wedge dy_j = \frac{i}{2} \sum dz_i \wedge d\bar{z}_i$]

For a Kähler manifold, $d\omega = 0$.

- Define. $L : \Omega^k \to \Omega^{k+2}$ by $L\alpha = \omega \wedge \alpha$

$\Omega^{r,s} \to \Omega^{r+1,s+1}$

Notice $[d, L]\alpha = d(L\alpha) - L(d\alpha) =$
$d(\omega \wedge \alpha) - \omega \wedge d\alpha$
$= \underbrace{d\omega}_{0} \wedge \alpha + \cancel{\omega \wedge d\alpha} - \cancel{\omega \wedge d\alpha}$
$= 0$.

$[d, L] = 0$

Let $\langle \alpha, \beta \rangle =$ inner product on diff forms $\alpha, \beta \in \Omega^k$.

- $\Lambda =$ Adjoint of L, ie. $\langle L \overset{k-2}{\alpha}, \overset{\in k}{\beta} \rangle = \langle \alpha, \Lambda \beta \rangle$

$\Lambda : \Omega^{r+1,s+1} \to \Omega^{r,s}$

$P^k = \ker(\Lambda|_{\Omega^k}) =$ "primitive k-forms".

Also • Let $H = \sum (k-n) \Pi^k$

Π^k is the projection onto the space of k-forms.

ie. $H(\overset{\text{k-form}}{\alpha}) = (k-n)\alpha$.

$\boxed{\text{Relations}}$ H, L, Λ satisfy

$[\Lambda, L] = H, [H, L] = -2L, [H, \Lambda] = +2\Lambda$

Gives a representation of $\mathfrak{sl}(2)$

Lefschetz decomposition $\Omega^k = \bigoplus_{i \geq 0} L^i (P^{k-2i})$

lots of other relations can be derived, such as $L^b H = H L^b + 2b L^b$
...

$\Rightarrow L^{n-k} : \Omega^k \to \Omega^{2n-k}$

is a bijection for $k \leq n$.

From before: $[d, L] = 0$, $d = \underset{d^{1,0}}{\partial} + \underset{d^{0,1}}{\bar\partial}$

$\Rightarrow$ • $[\partial, L] = 0 = (\partial L - L\partial)$

• $[\bar\partial, L] = 0$

Take adjoints. (L^2-adjoint : $\langle \alpha, \beta \rangle_{L^2} = \int_M \langle \alpha, \beta \rangle \, d\mathrm{vol}$)

$\Rightarrow$ • $[\Lambda, \partial^*] = 0$

$\Rightarrow$ • $[\Lambda, \bar\partial^*] = 0$

Question: $[\partial, \Lambda] = ?$

Homework for Loren:

• $[\partial, \Lambda] = -i\bar\partial^*$.

• $[\bar\partial, \Lambda] = i\partial^*$

adjoint

• $[L, \partial^*] = i\bar\partial$

• $[L, \bar\partial^*] = -i\partial$

Kähler Identities!

$$\text{Laplacian} = dd^* + d^*d = \Delta$$

$$= \sum -\frac{\partial^2}{\partial x_j^2} - \frac{\partial^2}{\partial y_j^2} \quad (\mathbb{R}^{2n})$$

$$\mathcal{H}^k = \{\alpha \in \Omega^k : \Delta\alpha = 0\} = \text{space of harmonic } k\text{-forms}$$

geometric analytic $\mathcal{H}^k \underset{\text{Hodge Theory}}{\cong} H^k(M)$ topological

Using the Kähler identities, we can show how Δ is related to

$$\Delta_\partial = \partial\partial^* + \partial^*\partial \text{ — } \ker \Delta_\partial^{r,s} \cong H_\partial^{r,s}(M)$$

same.

$$\Delta_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial \text{ —}$$

Relation with $\Delta = dd^* + d^*d$:

Automatic $d = \partial + \bar\partial \Rightarrow$ since $d^2 = 0$

$$\Rightarrow \underset{\text{cpx}}{(\partial+\bar\partial)^2} = \underset{\overset{\shortparallel}{0}}{\partial^2} + \underset{\overset{\shortparallel}{0}}{\bar\partial^2} + \underbrace{\partial\bar\partial + \bar\partial\partial}_{\{\partial,\bar\partial\}=0.} = 0$$

$$\Delta = (\partial+\bar\partial)(\partial^*+\bar\partial^*) + (\partial^*+\bar\partial^*)(\partial+\bar\partial)$$

$$= \underline{\partial\partial^* + \partial^*\partial} + \underline{\bar\partial\bar\partial^* + \partial^*\bar\partial}$$

$$+ \underline{\partial\bar\partial^* + \bar\partial^*\partial} + \bar\partial\partial^* + \partial^*\bar\partial$$

<u>Note:</u> $\underbrace{i(\partial\bar\partial^* + \bar\partial^*\partial)} = \partial[\partial,\Lambda] + [\partial,\Lambda]\partial$

$\left([\partial,\Lambda] = -i\bar\partial^*\right)$

$\Rightarrow i[\partial,\Lambda] = \bar\partial^*$

$$= \underset{0}{\partial^2\Lambda} - \cancel{\partial\Lambda\partial} + \cancel{\partial\Lambda\partial} - \Lambda\underset{0}{\partial^2}$$

$$= 0$$

$$\Rightarrow \Delta = \partial\partial^* + \partial^*\partial + \bar\partial\bar\partial^* + \bar\partial^*\partial$$

<u>Kähler mflds</u> $\Delta = \Delta_\partial + \Delta_{\bar\partial}$

note $[\Delta, d] = 0$ $\quad [\partial, \Delta_\partial] = 0$

$[\bar\partial, \Delta_{\bar\partial}] = 0$

We have $\Delta = \Delta_\partial + \Delta_{\bar\partial}$ $\quad \left(\partial^* = i[\Lambda,\bar\partial]\right)$

<u>Also:</u> $\Delta_\partial = \partial^*\partial + \partial\partial^*$

$$= i[\Lambda,\bar\partial]\partial + i\partial[\Lambda,\bar\partial]$$

$$= i\left[\Lambda\bar\partial\partial - \bar\partial\Lambda\partial + \partial\Lambda\bar\partial - \partial\bar\partial\Lambda\right]$$

$$= i\left[\Lambda\bar\partial\partial - \left(\bar\partial[\Lambda,\partial] + \bar\partial\partial\Lambda\right) + \left([\partial,\Lambda]\bar\partial + \Lambda\partial\bar\partial\right) - \partial\bar\partial\Lambda\right]$$

$[\partial,\Lambda] = -i\bar\partial^*$

$$= i\left[\cancel{\Lambda\bar\partial\partial} \underbrace{-i\,\bar\partial\bar\partial^*} \cancel{-\bar\partial\partial\Lambda} \underbrace{-i\,\bar\partial^*\bar\partial} + \cancel{\Lambda\partial\bar\partial}_{-\bar\partial\partial} \cancel{-\partial\bar\partial\Lambda}_{-\bar\partial\partial}\right]$$

$$\partial\bar\partial = -\bar\partial\partial$$

$$= \underbrace{\Delta_{\bar\partial}}$$

$$\Rightarrow \boxed{\Delta_\partial = \Delta_{\bar\partial}}.$$

<u>Further</u>, $[\Delta, L] = 0 = [\Delta, \Lambda]$

$$\Delta = \Delta_\partial + \Delta_{\bar\partial}$$

$$= 2\Delta_\partial = 2\Delta_{\bar\partial}$$

$$H^k(M) = \bigoplus_{r+s=k} H^{r,s}_{\bar\partial}(M)$$

Similar results are true for eigenspaces of the Laplacian.

$$E^{a,b}_{\lambda,\Delta} = \bigoplus_{k\geq 0} L^k P^{a-k,b-k}_{\lambda,\Delta} \qquad a+b$$

λ-eigenspace of Δ on (a,b) forms